JOURNAL OF APPROXIMATION THEORY 41, 297–308 (1984)

Best L_1-Approximation

HANS STRAUSS

Institut für Angewandte Mathematik, Universität Erlangen-Nürnberg, 8520 Erlangen, West Germany

Communicated by G. Meinardus

Received January 6, 1982

INTRODUCTION

Recently best L_1-approximation of continuous functions was extensively studied.

Galkin [2] and Strauss [10] showed that the problem of best approximation of continuous functions by polynomial splines has a unique solution. Micchelli [5] considered best L_1-approximation by weak Chebyshev subspaces and studied a class of functions for which the best approximation is the unique solution of an interpolation problem with nodes independent of the functions. DeVore [1] has established a very nice condition for unique one-sided best L_1-approximation which is very useful in applications to special functions.

In this paper we give first, a similar condition ensuring uniqueness of best L_1-approximation. This condition can be considered a generalized Haar condition. Using it, we give a short proof of uniqueness of best L_1-approximation from subspaces of spline functions.

Then we consider the relationship between best L_1-approximation and certain classes of perfect splines. We characterize best L_1-approximations from spline subspaces using perfect splines. We construct bounds for the error of best L_1-approximations from spline subspaces which satisfy certain boundary conditions. These estimates have an application in numerical integration.

1. UNIQUENESS IN L_1-APPROXIMATION

In this section we shall develop conditions ensuring that the best approximation of a continuous function in the L_1-norm is unique.

We shall need the following notation: Let $C[a,b]$ be the space of continuous real-valued functions on the interval $[a,b]$ normed by

0021-9045/84 $3.00

$\|f\| = \int_a^b |f(x)|\, dx$. If $f \in C[a, b]$ then $Z(f) = \{x \in [a, b] : f(x) = 0\}$, and two zeros x_1, x_2 of f are said to be *separated* if there is an x_0, $x_1 < x_0 < x_2$, such that $f(x_0) \neq 0$.

The following condition will turn out to be very important for uniqueness in L_1-approximation.

DEFINITION 1.1. Let $V = \text{span}\{v_1, \ldots, v_n\}$ be a subspace of $C[a, b]$ such that every function v in V has only a finite number of separated zeros. We say that the subspace V satisfies *condition* A, if there exists, for every nonzero v in V and every finite subset $Z_1 = \{t_1, \ldots, t_r\}$ of $Z(v) \cap (a, b)$, a nonzero w in V such that

(a) $(-1)^i w(x) \geqslant 0$ for $x \in [t_{i-1}, t_i]$, $i = 1, \ldots, r+1$, where $t_0 = a$, $t_{r+1} = b$;

(b) if v vanishes on an open subset of $[a, b]$, then w, too, vanishes there.

A similar condition concerning one-sided L_1-approximation was suggested by DeVore [1].

EXAMPLE 1.2. Every finite dimensional Chebyshev subspace of $C[a, b]$ satisfies condition A.

In Section 3 we show that subspaces of spline functions with fixed knots also satisfy condition A.

Condition A can be considered a generalized Haar condition.

Next we shall need the following results on L_1-approximation.

THEOREM 1.3. *Let $V = \text{span}\{v_1, \ldots, v_n\}$ be an n-dimensional subspace of $C[a, b]$ and f be a function in $C[a, b]$:*

(a) *The function v_0 in V is a best L_1-approximation from V to f, i.e., $\|f - v_0\| \leqslant \|f - v\|$ for all v in V, if and only if*

$$\left| \int_a^b v(x) \operatorname{sgn}(f - v_0)(x)\, dx \right| \leqslant \int_{Z(f - v_0)} |v(x)|\, dx$$

for all v in V.

(b) *Let v_1, v_2 be two best L_1-approximations then*

$$(f(x) - v_1(x))(f(x) - v_2(x)) \geqslant 0, \qquad x \in [a, b].$$

Proof. See Rice [8, pp. 104, 107].

THEOREM 1.4. *Let V be an n-dimensional subspace of $C[a, b]$ satisfying* *condition* A. *Then every function f in $C[a,b]$ has a unique best L_1-approximation from V.*

Proof. Let v_1, v_2 be two best approximations to f. We conclude from Theorem 1.3 that

$$(f(x) - v_1(x))(f(x) - v_2(x)) \geqslant 0, \qquad x \in [a,b].$$

W.l.o.g., we may assume that $v_2 = 0$. Then it is obvious that $(1/2)\, v_1$ is also a best approximation. Moreover, we have

$$|f(x) - (1/2)\, v_1(x)| = (1/2)\, |f(x) - v_1(x)| + (1/2)\, |f(x)|.$$

Hence it follows from $|f(x) - (1/2)\, v_1(x)| = 0$ that $|f(x) - v_1(x)| = |f(x)| = 0$. Therefore, $v_1(x) = 0$ if $f(x) - (1/2)\, v_1(x) = 0$.

Now we conclude from condition A that there exists a nonzero v_0 in V satisfying

$$(f(x) - (1/2)\, v_1(x))\, v_0(x) \geqslant 0, \qquad x \in [a,b],$$
$$(f(x_0) - (1/2)\, v_1(x_0))\, v_0(x_0) \neq 0$$

for some x_0 and the measure of the set $\{x : x \in Z(v_1),\ x \notin Z(v_0)\}$ is zero. Therefore

$$\left| \int_a^b \operatorname{sgn}(f(x) - (1/2)\, v_1(x))\, v_0(x)\, dx \right| > 0.$$

Moreover, it follows that

$$\int_{Z(f - (1/2) v_1)} |v_0(x)|\, dx = 0.$$

We conclude from Theorem 1.3 that $(1/2)\, v_1$ is not a best approximation. This contradiction proves the theorem.

2. Subspaces of Spline Functions

In this section we consider subspaces of spline functions with fixed knots satisfying certain boundary conditions.

Let S be the subspace of polynomial spline functions with fixed knots, $a < x_1 < x_2 < \cdots < x_m < b$, having multiplicities $m_1, ..., m_r$, respectively. Hence every s in S has the form

$$s(x) = \sum_{i=0}^{n-1} a_i x^i + \sum_{i=1}^{m} \sum_{j=1}^{m_i} b_{ij}(x - x_i)_+^{n-j}$$

$$\sum_{i=1}^{m} m_i = k, \qquad 0 < m_i \leqslant n.$$

Let $I \subset \{0,..., n-1\}$ and $J \subset \{0,..., n-1\}$ be two preassigned subsets of the set of indices $\{0,..., n-1\}$

$$I = \{i_r\}_{r=1}^{p}, \qquad J = \{j\}_{r=1}^{q}.$$

We denote by $S(C)$, where $C = \{I, J\}$ the subspace of S satisfying

$$S(C) = \{s \in S: s^{(i)}(a) = 0, i \in I,$$
$$s^{(j)}(b) = 0, j \in J\}.$$

Corresponding to I and J we define the sets

$$I' = \{n - i - 1: i \in \{0,..., n-1\} \backslash I\} = \{i'_r\}_{r=1}^{n-p},$$
$$J' = \{n - j - 1: j \in \{0,..., n-1\} \backslash J\} = \{j'_r\}_{r=1}^{n-q}.$$

It follows that $(I')' = I$ and $(J')' = J$.

We shall also need the following definitions.

A subset $\{v_1,..., v_n\}$ of linearly independent functions of $C[a, b]$ is called a *weak Chebyshev system* if every function v in $V = \text{span}\{v_1,..., v_n\}$ has at most $n-1$ sign changes on $[a, b]$. The subspace V is called a *weak Chebyshev subspace*. The subset $\{v_i\}_1^n$ is called a *complete weak Chebyshev system* if the subsets $\{v_i\}_1^k$ are weak Chebyshev systems for $k = 1,..., n$. The subspace V is called a *complete weak Chebyshev subspace* if V contains a basis $\{v_i\}_1^n$ which is a complete weak Chebyshev system.

It will be necessary that the sets I and J satisfy certain conditions.

Let I and J be subsets of $\{0,..., n-1\}$ such that

$$M_{v-1} + r \geqslant v, \qquad v = 1,..., n, \tag{2.1}$$

where $r = n + k - p - q \geqslant 0$ and M_v counts the number of terms in $\{i_1,..., i_p, j_1,..., j_q\}$ less than or equal to v.

LEMMA 2.1. *Let $S(C)$ be given where $C = \{I, J\}$ satisfy conditions* (2.1). *Then $S(C)$ is weak Chebyshev subspace of dimension $r = n + k - p - q$.*

Proof. This result is shown by Micchelli and Pinkus [7] for simple knots but it is also true for multiple knots.

LEMMA 2.2. *Suppose that I and J satisfy conditions* (2.1). *Then the corresponding subsets I' and J' satisfy*

$$M'_{\nu-1}+k\geqslant \nu, \qquad \nu=1,\dots,n, \tag{2.2}$$

where M'_ν counts the number of terms in $\{i'_1,\dots,i'_{n-p},j'_1,\dots,j'_{n-q}\}$ less than or equal to ν.

Proof. It can be shown that $M'_{\nu-1}=2\nu-p-q+M_{n-\nu-1}$, $\nu=1,\dots,n-1$, and it follows from the assumptions that $M_{n-\nu-1}+r\geqslant n-\nu$, $\nu=1,\dots,n-1$. Since $r=n+k-p-q$ we obtain $M'_{\nu-1}+k\geqslant \nu$, $\nu=1,\dots,n-1$. Moreover, $M'_{n-1}+k=2n-p-q+k=r+n$. This proves the lemma.

Henceforth it is always required that the boundary conditions C of $S(C)$ satisfy conditions (2.1).

3. UNIQUENESS IN L_1-APPROXIMATION FOR SUBSPACES OF SPLINE FUNCTIONS

Uniqueness in L_1-approximation for subspaces of spline functions has been shown in [2, 10]. Here we want to give a simple proof of uniqueness using the results of Section 1. Moreover, we consider approximation problems satisfying certain boundary conditions. These results can be applied to numerical integration.

LEMMA 3.1. *Let V be an n-dimensional weak Chebyshev subspace of $C[a,b]$.*

(a) *Then V is a complete weak Chebyshev subspace.*

(b) *Given $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$. Then there exists a nontrivial v in V such that*

$$(-1)^{i+1}v(x)\geqslant 0, \qquad x_{i-1}<x<x_i, \qquad i=1,\dots,n.$$

Proof. (a) See Sommer and Strauss [9].

(b) See Jones and Karlovitz [3].

Now we shall show the following result:

THEOREM 3.2. *The subspace $S(C)$ satisfies condition* A.

Proof. Let s be a function of $S(C)$. We shall distinguish the following cases:

(a) Suppose that s has no zero interval. According to Lemma 2.1, $S(C)$ is a weak Chebyshev subspace. It follows from Lemma 3.1(a) that

$S(C)$ is a complete weak Chebyshev subspace. Assume that s has the zeros $U = \{u_i\}_1^t$, $t < r$, on (a, b). Let $U_1 = \{v_i\}_1^\nu$ be a subset of U. Then there exists a weak Chebyshev subspace of dimension $\nu + 1$ of $S(C)$. It follows from Lemma 3.1 that there is a nontrivial $s_1 \in S(C)$

$$(-1)^i s_1(x) \geqslant 0, \qquad x \in [v_{i-1}, v_i], \qquad i = 1,..., \nu + 1,$$
$$v_0 = a, \quad v_{\nu+1} = b.$$

(b) The function s has a zero interval. Assume that $[x_i, x_j]$ is a subinterval such that $s(x) = 0$, $x \in [x_i, x_j]$ and s has no zero interval on $[a, x_i]$. We define

$$V_1 = \{s|_{[a,x_i]} : s \in S, s^{(i)}(a) = 0, i \in I,$$
$$s^{(j)}(x_i) = 0, j \in 0,..., m - m_i - 1\}.$$

Let $U = \{u_i\}_1^t$ be the set of zeros of s on (a, x_i) and let $\{v_i\}_1^\nu$ be a subset of U. It follows from (a) that there exists some $\bar{s} \in V_1$ satisfying

$$(-1)^i \bar{s}(x) \geqslant 0, \qquad x \in [v_{i-1}, v_i], \qquad i = 1,..., \nu + 1,$$
$$v_0 = x_0, \quad v_{\nu+1} = x_i.$$

Then we define $s_1 \in S(C)$ by $s_1(x) = \bar{s}(x)$, $x \in [a, x_i)$ and $s_1(x) = 0$ elsewhere.

(c) Similarly a function s_1 can be constructed if $x_i = a$, $x_j < b$.

This proves that $S(C)$ satisfies condition A.

Now we shall study an approximation problem satisfying boundary conditions. Let the subspace S in $C[a, b]$ be given. Let $C = \{I, J\}$ be boundary conditions satisfying (2.1). Suppose that g in $C[a, b]$ is a function such that $g^{(i_p)}(a)$ exists if $I \neq \varnothing$ and $g^{(j_q)}(b)$ exists if $J \neq \varnothing$. We define

$$V_g(C) = \{s \in S: s^{(i)}(a) = g^{(i)}(a), i \in I,$$
$$s^{(j)}(b) = g^{(j)}(b), j \in J\}.$$

It is well-known that $V_g(C) \neq \varnothing$ if I and J satisfy (2.1).

THEOREM 3.3. *There is a unique function* s_0 *in* $V_g(C)$ *satisfying* $\| g - s_0 \| \leqslant \| g - s \|$ *for all* $s \in V_g(C)$.

Proof. Let $\bar{s}$ be a function of $V_g(C)$. We define $f = g - \bar{s}$. It follows from Theorem 1.4 and Theorem 3.2 that there exists a unique best L_1-approximation s_1 from $S(C)$ to f. Hence $s_0 = \bar{s} + s_1$ is the unique function satisfying $\| g - s_0 \| \leqslant \| g - s \|$, $s \in V_g(C)$.

Finally we shall give an example that unicity is not true for weak Chebyshev subspaces, in general.

EXAMPLE 3.4. We define the following functions on $[0, 5]$. Let

$$v_1(x) = 1,$$

$$v_2(x) = \begin{cases} -1 + x & 0 \leqslant x < 1 \\ 0 & 1 \leqslant x \leqslant 4, \\ x - 4 & 4 < x \leqslant 5 \end{cases}$$

$$f(x) = \begin{cases} 1 - x & 0 \leqslant x < 1 \\ 0 & 1 \leqslant x \leqslant 4 \\ x - 4 & 4 < x \leqslant 5 \end{cases}$$

Then every function of $V = \operatorname{span}\{v_1, v_2\}$ has at most one sign change, i.e., V is a weak Chebyshev subspace. On the other hand, we have for $0 \leqslant c < 1$ that

$$\int_0^5 v_2(x) \operatorname{sgn}(f - cv_2)(x)\, dx = 0,$$

$$2a = \left| \int_0^5 v_1(x) \operatorname{sgn}(f - cv_2)(x)\, dx \right| < \int_{Z(f - cv_2)} |v_1(x)| dx = 3.$$

Hence we conclude from Theorem 1.3 that cv_2, $0 \leqslant c < 1$, are best approximations to f.

4. L_1-APPROXIMATION AND PERFECT SPLINES

In this section we shall show that best L_1-approximation for subspaces of spline functions is closely related to certain classes of perfect splines.

A *perfect spline* of degree n $(n \geqslant 1)$ with r knots on $[a, b]$ is a function of the form

$$P(x) = c\left(x^n + 2 \sum_{i=1}^{r} (-1)^i (x - u_i)_+^n\right) + \sum_{j=0}^{n-1} a_j x^j \tag{4.1}$$

where $c, a_0, ..., a_{n-1}$ are real constants and the knots $\{u_i\}$ satisfy $a < u_1 < \cdots < u_r < b$.

Let S be the subspace of Section 2. Suppose that f and s are two functions such that $f \in C^{(n)}[a, b]$ and $s \in S$.

Repeated integration by parts yields the identity

$$\int_a^b (f-s)(x)\, P^{(n)}(x)\, dx$$

$$= \sum_{i=0}^{n-1} (-1)^i (f-s)^{(i)}(x)\, P^{(n-i-1)}(x) \Big|_a^b$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{m_i} (-1)^{n-j} (f-s)^{(n-j)}(x)\, P^{(j-1)}(x) \Big|_{x_i+0}^{x_i-0}$$

$$+ (-1)^n \int_a^b f^{(n)}(x)\, P(x)\, dx. \tag{4.2}$$

The following class of perfect splines will be very important.

Let $S(C)$ be a subspace where $C = \{I, J\}$ satisfies (2.1). Suppose that $C' = \{I', J'\}$ are the boundary conditions corresponding to C. Let $P(C')$ be the set of perfect splines of degree n satisfying

$$P^{(i)}(a) = 0, \qquad i \in I', \qquad P^{(j)}(b) = 0, \qquad j \in J',$$
$$P^{(j)}(x_i) = 0, \qquad j = 0, \ldots, m_i - 1 \qquad \text{for} \quad i = 1, \ldots, m, \tag{4.3}$$
$$|P^{(n)}(x)| = 1.$$

It is said that the *class* $P(C')$ *corresponds* to $S(C)$.

LEMMA 4.1. *There exists a perfect spline P in $P(C')$ with at most $n + k - p - q$ knots.*

Proof. See Karlin [4].

LEMMA 4.2. *Let h be a function such that $h(x) = \varepsilon(-1)^i$ a.e. on (t_{i-1}, t_i), where $a = t_0 < t_1 < \cdots < t_{r+1} = b$, $i = 1, \ldots, r+1$ and $\varepsilon \in \{-1, 1\}$ satisfying*

$$\int_a^b s(x)\, h(x)\, dx = 0$$

for all $s \in S(C)$. Then there exists a perfect spline $P \in P(C')$ satisfying $P^{(n)} = h$ a.e.

Proof. Let $P(x)$ be of the form (4.1) where $t_i = u_i$, $i = 1, \ldots, r$. Then we determine the coefficients $a_0, \ldots, a_{n-1}$ of P such that the first n conditions of (4.3) are satisfied. Since the boundary conditions C' satisfy (2.2) the set $\{a_i\}$ is uniquely determined. Then it is possible to prove with identity (4.3) where $f \equiv 0$ that the other conditions of (4.3) are also satisfied. (See also [11, Theorem 2.3].)

Henceforth we shall always consider a function g and a subset $V_g(C)$ satisfying the properties of Section 3.

THEOREM 4.3. *Let s_0 be a function in $V_g(C)$ such that $g - s_0$ vanishes only on a set of measure zero. Then s_0 is a best L_1-approximation to g out of $V_g(C)$ if and only if there exists a perfect spline P in $P(C')$ where $P(C')$ corresponds to $S(C)$ such that $P^{(n)} = \operatorname{sgn}(g - s_0)$ a.e.*

Proof. (a) Let s_0 be a best approximation. Then it follows from Theorem 1.3 that

$$\int_a^b s(x) \operatorname{sgn}(g(x) - s_0(x))\, dx = 0, \qquad s \in S(C).$$

Hence it follows from Lemma 4.2 that there exists a P in $P(C')$ satisfying $P^{(n)} = \operatorname{sgn}(g - s_0)$ a.e.

(b) There is a P (C') such that $P^{(n)} = \operatorname{sgn}(g - s_0)$ a.e. Let $f \equiv 0$ in (4.2). Then it follows from this identity that

$$\int_a^b s(x)\, P^{(n)}(x)\, dx = 0, \qquad s \in S(C).$$

We conclude from Theorem 1.3 and $P^{(n)} = \operatorname{sgn}(g - s_0)$ a.e. that 0 is a best approximation from $S(C)$ to $g - s_0$. Hence s_0 is a best approximation to g out of $V_g(C)$.

THEOREM 4.4. *Let s_0 be a best L_1-approximation to g in $C^{(n)}[a, b]$ out of $V_g(C)$. Suppose that $g - s_0$ vanishes only on a set of measure zero.*

(a) *Then*

$$\| g - s_0 \| \geqslant \left| \int_a^b g^{(n)}(x)\, P(x)\, dx \right|$$

for all P in $P(C')$.

(b) *There exists a P_0 in $P(C')$ such that*

$$\| g - s_0 \| = \left| \int_a^b g^{(n)}(x)\, P_0(x)\, dx \right|.$$

Proof. (a) It follows from $|P^{(n)}(x)| = 1$ that

$$\int_a^b | g(x) - s_0(x) |\, dx \geqslant \left| \int_a^b (g(x) - s_0(x))\, P^{(n)}(x)\, dx \right|$$

for all P in $P(C')$. Then we conclude from (4.2) that

$$\left| \int_a^b (g(x) - s_0(x))\, P^{(n)}(x)\, dx \right| = \left| \int_a^b g^{(n)}(x)\, P(x)\, dx \right|.$$

(b) Let P_0 be the perfect spline of Theorem 4.3 corresponding to the best approximation s_0. Then

$$\int_a^b | g(x) - s_0(x)|\, dx = \left| \int_a^b (g(x) - s_0(x))\, P_0^{(n)}(x)\, dx \right|.$$

Now the assertion follows as in (a).

5. Estimates

In this section we want to estimate the error in L_1-approximation for some special problems. The estimates have an application in numerical integration.

THEOREM 5.1. *Let g in $C^{(n)}[a, b]$ and $V_g(C)$ be given. Suppose that s_0 is a best L_1-approximation to g out of $V_g(C)$ and $g - s_0$ vanishes only on a set of measure zero. Then there exists a P_0 in $P(C')$ satisfying*

$$\| g - s_0\| \leqslant \|P_0\|_\infty \, \| g^{(n)} \|$$

where $\|P_0\|_\infty = \max_{a \leqslant x \leqslant b} |P_0(x)|$.

Proof. We conclude from Theorem 4.4(b) that there is a perfect spline P_0 in $P(C')$ such that

$$\| g - s_0\| = \left| \int_a^b g^{(n)}(x)\, P_0(x)\, dx \right|.$$

Hence

$$\| g - s_0\| \leqslant \|P_0\|_\infty \, \| g^{(n)} \|.$$

Micchelli, Rivlin and Winograd [6] proved the following result.

LEMMA 5.2. *Let P_0 be a perfect spline of degree n satisfying $P(x_i) = 0$, $i = 0,..., k + 1$ for $0 = x_0 < x_1 < \cdots < x_{k+1}$. Then*

$$\|P\|_\infty \leqslant k_i \|P^{(n)}\|_\infty, \qquad i = 1, 2$$

where

$$k_1 = \Delta^n/4n, \qquad k_2 = ((n-1)^{n-1}/n!)(\Delta/2)^n \tag{5.1}$$

and $\Delta = \max_{i=0,\ldots,k} \{|x_{i+1} - x_i|\}$.

THEOREM 5.3. *Let g in $C^{(n)}[0,1]$ be given. Suppose that $V_g(C) = \{s \in S: (g-s)^{(i)}(0) = (g-s)^{(i)}(1) = 0, \quad i = 0,\ldots, n-2\}$, where $0 = x_0 < x_1 < \cdots < x_k < x_{k+1} = 1$ are the knots of S. Suppose that s_0 is a best L_1-approximation to g out of $V_g(C)$ and $g - s_0$ vanishes only on a set of measure zero. Then*

$$\| g - s_0 \| \leqslant k_i \| g^{(n)} \|, \qquad i = 1, 2$$

where k_1, k_2 are defined in (5.1).

Proof. We conclude from Theorem 5.1 that there exists a perfect spline P_0 satisying $P_0(x_i) = 0$, $i = 0,\ldots, k+1$, $\|P_0^{(n)}\|_\infty = 1$ and

$$\| g - s_0 \| \leqslant \|P_0\|_\infty \| g^{(n)} \|.$$

Then the assertions follow from Lemma 5.2.

Remark. These results can be applied in numerical integration. The relationship between L_1-approximation with spline functions and best quadrature formulae was studied by Strauss [11] in detail. Using the theorems of this section we are able to give estimates for the remainder functionals of these best quadrature formulae.

REFERENCES

1. R. DEVORE, private communication.
2. R. V. GALKIN, The uniqueness of the element of best mean approximation to a continuous function using splines with fixed nodes, *Math. Notes* **15** (1974), 3–8.
3. R. C. JONES AND L. A. KARLOVITZ, Equioscillation under nonuniqueness in the approximation of continuous functions, *J. Approx. Theory* **3** (1970), 138–145.
4. S. KARLIN, Interpolation properties of generalized perfect splines and the solution of certain extremal problems, I, *Trans. Amer. Math. Soc.* **206** (1975), 25–66.
5. C. A. MICCHELLI, Best L^1 approximation by weak Chebyshev systems and the uniqueness of interpolating perfect splines, *J. Approx. Theory* **19** (1977), 1–14.
6. C. A. MICCHELLI, T. J. RIVLIN, AND S. WINOGRAD, The optimal recovery of smooth functions, *Numer. Math.* **26** (1976), 191–200.
7. C. A. MICCHELLI AND A. PINKUS, Moment theory for weak Chebyshev systems with applications to monosplines, quadrature formulae and best one-sided L^1-approximation by spline functions with fixed knots, *SIAM J. Math. Anal* **8** (1977), 206–230.
8. J. R. RICE, "The Approximation of Functions," Vol. I, Addison–Wesley, Reading, Mass., 1964.

9. M. SOMMER AND H. STRAUSS, Eigenschaften von schwach tschebyscheffschen Räumen, *J. Approx. Theory* **21** (1977), 257–268.
10. H. STRAUSS, L_1-Approximation mit Splinefunktionen, *in* "Numerische Methoden der Approximationstheorie, Oberwolfach 1973" (L. Collatz and G. Meinardus, Eds.), pp. 151–162, ISNM 26, Birkhäuser-Verlag, Basel, 1975.
11. H. STRAUSS, Approximation mit Splinefunktionen und Quadraturformeln. *In* "Spline Functions, Karlsruhe 1975" (K. Böhmer, G. Meinardus, and W. Schempp, Eds.), pp. 371–419, Lecture Notes in Mathematics No. 501, Springer-Verlag, Berlin, 1976.